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Bounding the number of odd paths in planar graphs via convex optimization. (English. English summary)

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A classical topic in extremal graph theory is the study of the extremal function $\mathrm{ex}(n, H, \mathcal{F})$, which denotes the maximum number of copies of the graph H in an n-vertex graph G that is F-free for every graph F in the collection $\mathcal{F}$. Turán's theorem [P. Turán, Mat. Fiz. Lapok **48** (1941), 436–452; MR0018405], which launched this area of study, considers the case when $H = K_2$, and $\mathcal{F} = \{K_{r+1}\}$, $r \geq 2$.

A variation on this theme is to consider geometric, instead of combinatorial, restrictions on the graphs G. So, let $N_{\mathcal{P}}(n, H)$ denote the maximum number of copies of the graph H in a planar graph G on n vertices. S. L. Hakimi and E. F. Schmeichel [J. Graph Theory **3** (1979), no. 1, 69–86; MR0519175] initiated the study of this function; specifically, they looked at the case when H is a cycle C_k, $k \geq 3$. N. Alon and Y. Caro [in *Convexity and graph theory (Jerusalem, 1981)*, 25–36, North-Holland Math. Stud., 87, North-Holland, Amsterdam, 1984; MR0791009] studied the case when H is a complete bipartite graph $K_{1,k}$ or $K_{2,k}$, $k \geq 2$. In this paper, the authors consider the case when H is an odd path P_{2m+1}, $m \geq 2$.

It is known from the work of T. Huynh, G. Joret and D. R. Wood [Combin. Probab. Comput. **31** (2022), no. 5, 812–839; MR4472290] that $N_{\mathcal{P}}(n, P_{2m+1}) = \Theta(n^{m+1})$. Alon and Caro and D. Ghosh et al. [Discrete Math. **344** (2021), no. 5, Paper No. 112317; MR4218486] determined $N_{\mathcal{P}}(n, P_{2m+1})$ for $m = 1$ and $m = 2$, respectively. Ghosh et al. also conjectured that $N_{\mathcal{P}}(n, P_{2m+1}) = (4m^{-m} + o(1))n^{m+1}$ for all $m \geq 2$. C. Cox and R. R. Martin [J. Graph Theory **101** (2022), no. 3, 521–558; MR4512144] proved that $N_{\mathcal{P}}(n, P_{2m+1}) \leq (\rho(m)/2 + o(1))n^{m+1}$ for all $m \geq 2$, where $\rho(m)$ is the solution to a certain convex optimization problem. Furthermore, they conjectured that $\rho(m) \leq 8m^{-m}$.

In this paper, the authors prove the conjectures of Cox and Martin and Ghosh et al. up to constant factors: specifically, they show that for each fixed $m \geq 2$, for large enough n we have that $N_{\mathcal{P}}(n, P_{2m+1}) \leq 10^4 m^{-m} n^{m+1}$. Analogous to the work of Cox and Martin, the authors first compute bounds on a quantity $\beta(P_m)$, which, roughly speaking, maximizes the probability of hitting a copy of P_m when $m-1$ edges of a complete graph are chosen independently with respect to any probability measure on the edges of the complete graph. Then, they employ a weight-shifting argument to transfer any bound on $\beta(P_m)$ into a bound on $\rho(m)$. No attempt is made to optimize the constants.

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.